Fourier Analysis

Mar 21, 2024

Review.

Def. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be of moderate decrease if

① f is cts on $\mathbb{R}$.

② $\exists\ A > 0$ such that

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

Write $M(\mathbb{R}) = \{ f : f \text{ is of moderate decrease} \}$.

Def. (improper integration)

Let $f \in M(\mathbb{R})$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx.$$

$$= \lim_{N,M\to+\infty} \int_{-N}^{M} f(x)\,dx$$

§ 5.2 Fourier transform on $\mathbb{R}$.

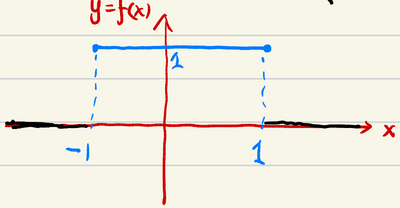
Def. Let $f \in \mathcal{M}(\mathbb{R})$. The Fourier transform of f is

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i \xi x} \, dx, \quad \xi \in \mathbb{R}.$$

Here: $\hat{f} : \mathbb{R} \to \mathbb{C}$.

- $f(x) e^{-2\pi i \xi x} \in \mathcal{M}(\mathbb{R})$ for every $\xi \in \mathbb{R}$.
- $|\hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \right|$
  $\leq \int_{-\infty}^{\infty} |f(x)| \, dx < \infty.$

Example 1. Let $f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$



For $\xi \neq 0$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

$$= \int_{-1}^{1} 1 \cdot e^{-2\pi i \xi x} dx$$

$$= \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Bigg|_{x=-1}^{1}$$

$$= \frac{e^{-2\pi i \xi} - e^{2\pi i \xi}}{-2i \, (\pi \xi)}$$

$$= \frac{\sin(2\pi \xi)}{\pi \xi}.$$

If $\xi = 0$, we have

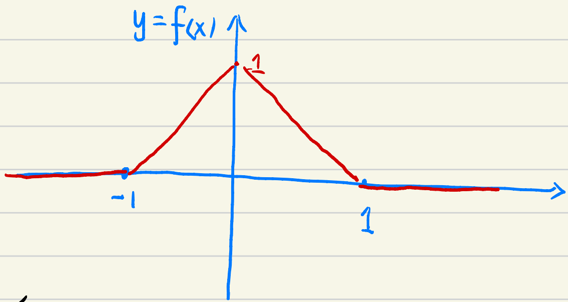
$$\hat{f}(0) = \int_{-1}^{1} 1 \, dx = 2$$

So

$$\hat{f}(\xi) = \begin{cases} \dfrac{\sin(2\pi\xi)}{\pi\xi} & \text{if } \xi \neq 0 \\ 2 & \text{if } \xi = 0. \end{cases}$$

Example 2. Let

$$f(x) = \begin{cases} 1-|x| & \text{if} \quad |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$



For $\xi \neq 0$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-1}^{1} (1-|x|)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{0}^{1} (1-x)\, e^{-2\pi i \xi x}\, dx + \int_{-1}^{0} (1+x)\, e^{-2\pi i \xi x}\, dx$$

$$= (1-x)\, \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Bigg|_0^1 - \int_0^1 \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \cdot \underline{(-1)}\, dx$$

$$(1+x)\frac{e^{-2\pi i\xi x}}{-2\pi i\xi}\Bigg|_{-1}^{0} - \int_{-1}^{0} \frac{e^{-2\pi i\xi x}}{-2\pi i\xi}\cdot 1\, dx$$

$$= \cancel{\frac{1}{2\pi i\xi}} + \frac{e^{-2\pi i\xi x}}{(-2\pi i\xi)^2}\Bigg|_{0}^{1}$$

$$+ \cancel{\frac{1}{-2\pi i\xi}} - \frac{e^{-2\pi i\xi x}}{(-2\pi i\xi)^2}\Bigg|_{-1}^{0}$$

$$= \frac{e^{-2\pi i\xi} - 1}{-4\pi^2\xi^2} - \frac{1 - e^{2\pi i\xi}}{-4\pi^2\xi^2}$$

$$= \frac{2 - e^{-2\pi i\xi} - e^{2\pi i\xi}}{4\pi^2\xi^2}$$

$$= \frac{2 - 2\cos(2\pi\xi)}{4\pi^2\xi^2} = \frac{\sin^2(\pi\xi)}{\pi^2\xi^2}$$

when $\xi = 0$,

$$\hat{f}(0) = \int_{-1}^{1} (1-|x|)\,dx = 1.$$

§ 5.3. Some basic properties of Fourier transform.

Prop 1. Let $f \in \mathcal{M}(\mathbb{R})$. Then the following hold:

(1) $f(x+h) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot e^{2\pi i \xi h} \qquad \forall\, h \in \mathbb{R}.$

(2) $f(x) \cdot e^{-2\pi i h x} \xrightarrow{\mathcal{F}} \hat{f}(\xi + h), \qquad \forall\, h \in \mathbb{R}.$

(3) Let $\delta > 0$. Then

$$f(\delta x) \xrightarrow{\mathcal{F}} \frac{\hat{f}\left(\frac{\xi}{\delta}\right)}{\delta}.$$

(4) Suppose $f' \in \mathcal{M}(\mathbb{R})$. Then

$$f'(x) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot (2\pi i \xi)$$

(5) Suppose $x f(x) \in \mathcal{M}(\mathbb{R})$. Then

$$f(x) \cdot (-2\pi i x) \xrightarrow{\mathcal{F}} \frac{d \hat{f}(\xi)}{d \xi}$$

Pf. We first prove (1).

For $h \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(x+h)\, e^{-2\pi i \xi x}\, dx$$

$$= \lim_{N\to\infty} \int_{-N}^{N} f(x+h)\, e^{-2\pi i \xi x}\, dx$$

Letting $y=x+h$

$$= \lim_{N\to\infty} \int_{-N+h}^{N+h} f(y)\, e^{-2\pi i \xi (y-h)}\, dy$$

$$= \lim_{N\to\infty} e^{2\pi i \xi h} \int_{-N+h}^{N+h} f(y)\, e^{-2\pi i \xi y}\, dy$$

$$= e^{2\pi i \xi h}\, \hat{f}(\xi).$$

The proofs of ② & ③ are similar.

Now we prove ④:

$$\int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i \xi x}\, dx$$

(Using the assumption $f' \in M(\mathbb{R})$)

$$= \lim_{N\to\infty} \int_{-N}^{N} f'(x)\, e^{-2\pi i \xi x}\, dx$$

(integration by parts)

$$= \lim_{N\to\infty} \left(f(x)\, e^{-2\pi i \xi x} \Big|_{-N}^{N} - \int_{-N}^{N} f(x) \cdot (-2\pi i \xi) \cdot e^{-2\pi i \xi x}\, dx \right)$$

$$= \lim_{N\to\infty} (2\pi i \xi) \int_{-N}^{N} f(x)\, e^{-2\pi i \xi x}\, dx.$$

$$= (2\pi i \xi)\, \hat{f}(\xi).$$